

THE REES VALUATIONS OF COMPLETE IDEALS IN A REGULAR LOCAL RING

WILLIAM HEINZER AND MEE-KYOUNG KIM

ABSTRACT. Let I be a complete \mathfrak{m} -primary ideal of a regular local ring (R, \mathfrak{m}) of dimension $d \geq 2$. In the case of dimension two, the beautiful theory developed by Zariski implies that I factors uniquely as a product of powers of simple complete ideals and each of the simple complete factors of I has a unique Rees valuation. In the higher dimensional case, a simple complete ideal of R often has more than one Rees valuation, and a complete \mathfrak{m} -primary ideal I may have finitely many or infinitely many base points. For the ideals having finitely many base points Lipman proves a unique factorization involving special $*$ -simple complete ideals and possibly negative exponents of the factors. Let T be an infinitely near point to R with $\dim R = \dim T$ and $R/\mathfrak{m} = T/\mathfrak{m}_T$. We prove that the special $*$ -simple complete ideal P_{RT} has a unique Rees valuation if and only if either $\dim R = 2$ or there is no change of direction in the unique finite sequence of local quadratic transformations from R to T . We also examine conditions for a complete ideal to be projectively full.

1. INTRODUCTION

Motivation for our work in this paper comes from an interesting article of Joseph Lipman [L]. Lipman considers the structure of a certain class of complete ideals, the finitely supported complete ideals, in a regular local ring (RLR) of dimension $d \geq 2$. He proves a factorization theorem for the finitely supported complete ideals that extends the factorization theory of complete ideals in a two-dimensional RLR as developed by Zariski [ZS2, Appendix 5]. Other work on this topic has been done by John Gately in [G1] and [G2], and by Campillo, Gonzalez-Sprinberg and Lejeune-Jalabert in [CGL].

All rings we consider are assumed to be commutative with an identity element. We use the concept of complete ideals as defined and discussed in Swanson-Huneke [SH, Chapters 5,6,14]. We also use a number of concepts considered in Lipman's paper [L]. The product of two complete ideals in a two-dimensional regular local

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ring is again complete. This no longer holds in higher dimension, [C] or [Hu]. To consider the higher dimensional case, one defines for ideals I and J the $*$ -product, $I * J$ to be the completion of IJ . A complete ideal I in a commutative ring R is said to be **$*$ -simple** if $I \neq R$ and if $I = J * L$ with ideals J and L in R implies that either $J = R$ or $L = R$.

Another concept used by Zariski in [ZS2] is that of the transform of an ideal; the complete transform of an ideal is used in [L] and [G2].

Definition 1.1. Let $R \subseteq T$ be unique factorization domains (UFDs) with R and T having the same field of fractions, and let I be an ideal of R not contained in any proper principal ideal.

- (1) The **transform** of I in T is the ideal $I^T = a^{-1}IT$, where aT is the smallest principal ideal in T that contains IT .
- (2) The **complete transform** of I in T is the completion $\overline{I^T}$ of I^T .

A proper ideal I in a commutative ring R is **simple** if $I \neq L \cdot H$, for any proper ideals L and H . An element $\alpha \in R$ is said to be **integral over** I if α satisfies an equation of the form

$$\alpha^n + r_1\alpha^{n-1} + \cdots + r_n = 0, \quad \text{where } r_i \in I^i.$$

The set of all elements in R that are integral over an ideal I forms an ideal, denoted by \overline{I} and called the **integral closure** of I . An ideal I is said to be **complete** (or, **integrally closed**) if $I = \overline{I}$.

For an ideal I of a local ring (R, \mathfrak{m}) , the **order** of I , denoted $\text{ord}_R I$, is r if $I \subseteq \mathfrak{m}^r$ but $I \not\subseteq \mathfrak{m}^{r+1}$. If (R, \mathfrak{m}) is a regular local ring, the function that associates to an element $a \in R$, the order of the principal ideal aR , defines a discrete rank-one valuation, denoted ord_R on the field of fractions of R . The associated valuation ring (DVR) is called **the order valuation ring** of R .

Let I be a nonzero ideal of a Noetherian integral domain R . The set of **Rees valuation rings** of I is denoted **Rees** I , or by **Rees** $_R I$ to also indicate the ring in which I is an ideal. It is by definition the set of DVRs

$$\left\{ \left(\overline{R \left[\frac{I}{a} \right]} \right)_Q \mid 0 \neq a \in I \text{ and } Q \in \text{Spec} \left(\overline{R \left[\frac{I}{a} \right]} \right) \text{ is of height one with } I \subset Q \right\},$$

where $\overline{}$ denotes integral closure in the field of fractions. The corresponding discrete valuations with value group \mathbb{Z} are called the **Rees valuations** of I . In general, if $J \subseteq I$ is a reduction, then we have $\text{Rees } J = \text{Rees } I$.

An ideal I is said to be **normal** if all the powers of I are complete. Let I be a normal \mathbf{m} -primary ideal of a normal Noetherian local domain (R, \mathbf{m}) . The minimal prime ideals of $\mathbf{m}R[It]$ in the Rees algebra $R[It]$ are in one-to-one correspondence with the Rees valuation rings of I . The correspondence associates to each Rees valuation ring V of I a unique prime $P \in \text{Min}(\mathbf{m}R[It])$ such that $V = R[It]_P \cap \mathcal{Q}(R)$. Properties of the quotient ring $R[It]/P$ relate to properties of certain birational extensions of R .

If (R, \mathbf{m}) is a two-dimensional regular local ring, then the Zariski theory implies that a simple complete \mathbf{m} -primary ideal has a unique Rees valuation ring. However, if the dimension of R is greater than two, then a simple complete \mathbf{m} -primary ideal may have more than one Rees valuation ring; indeed, this is often the case even for a special $*$ -simple complete ideal as in Definition 2.8. An ideal I of a Noetherian integral domain R is said to be **one-fibered** if I has a unique Rees valuation.

In the case where (R, \mathbf{m}) is a two-dimensional regular local ring, Zariski's unique factorization theorem implies that a complete \mathbf{m} -primary ideal I can be factored uniquely as a finite product of powers of simple complete ideals. The distinct simple factors of I are in one-to-one correspondence with the Rees valuation rings of I .

If I is a simple complete ideal of a two-dimensional RLR and R/\mathbf{m} is algebraically closed, Huneke and Sally [HS, Theorem 3.8] prove that $R[It]/P$ is regular. This result is extended in [K, Theorem 3.1] by proving that if R/\mathbf{m} is algebraically closed, then I is a product of distinct simple complete ideals if and only if $R[It]/P$ is regular for each minimal prime P of $\mathbf{m}R[It]$.

Let (R, \mathbf{m}) be a regular local ring of dimension $d \geq 2$. In Section 2 we discuss the structure of regular local rings T birational over R , and the order valuation ring of T . In Section 3 we review Lipman's unique factorization theorem and raise several questions about the base points of finitely supported complete ideals. Let I be an \mathbf{m} -primary ideal of R . In Section 4 we compare the Rees valuations of I with the Rees valuations of the transform I_1 of I in $S_1 = R[\frac{\mathbf{m}}{x}]$, where $x \in \mathbf{m} \setminus \mathbf{m}^2$. We prove in Proposition 4.3 that $\text{Rees } I \subseteq \text{Rees}_{S_1} I_1 \cup \text{Rees } \mathbf{m}$. If I is finitely supported, we prove in Proposition 4.6 that $\text{Rees}_{S_1} I_1 \subseteq \text{Rees } I$, and demonstrate in Example 4.9 that this may fail if I is not finitely supported.

We observe in Remark 5.2 that every special $*$ -simple complete ideal is projectively full. In Proposition 5.6 we prove that a complete \mathbf{m} -primary ideal of R is projectively full if the transform I_1 of I in S_1 is projectively full. In Section 6 we examine

the structure of special $*$ -simple complete ideals in terms of their Rees valuations. Let T be an infinitely near point to R with $\dim R = \dim T$ and $R/\mathbf{m} = T/\mathbf{m}_T$. We prove in Theorem 6.8 that the special $*$ -simple complete ideal P_{RT} has a unique Rees valuation if and only if either $\dim R = 2$ or there is no change of direction in the unique finite sequence of local quadratic transformations from R to T . In the case where $T = R_1$ is a first local quadratic transform of R and $R/\mathbf{m} \neq T/\mathbf{m}_T$, we demonstrate in Examples 6.11 and 6.12 that sometimes the special $*$ -simple complete ideal P_{RT} has two Rees valuations and sometimes only one Rees valuation. Examples 6.13 and 6.14 illustrate a pattern where from R_0 to R_2 or from R_0 to R_3 there is exactly one or exactly two changes of direction.

2. PRELIMINARIES

Let V be a valuation domain and let R be a subring of V . Let $\mathbf{m}(V)$ denote the unique maximal ideal of V . We call the prime ideal $\mathbf{m}(V) \cap R$ of R **the center** of V on R .

Let (R, \mathbf{m}) be a Noetherian local domain with field of fractions $\mathcal{Q}(R)$. A valuation domain $(V, \mathbf{m}(V))$ is said to **birationally dominate** R if $R \subseteq V \subseteq \mathcal{Q}(R)$ and $\mathbf{m}(V) \cap R = \mathbf{m}$, that is, \mathbf{m} is the center of V on R . The valuation domain V is said to be a **prime divisor** of R if V birationally dominates R and the transcendence degree of the field $V/\mathbf{m}(V)$ over R/\mathbf{m} is $\dim R - 1$. If V is a prime divisor of R , then V is a DVR [A, p. 330].

The **quadratic dilatation** or **blowup** of \mathbf{m} along V , cf. [N, page 141], is the unique local ring on the blowup $\text{Bl}_{\mathbf{m}}(R)$ of \mathbf{m} that is dominated by V . The ideal $\mathbf{m}V$ is principal and is generated by an element of \mathbf{m} . Let $a \in \mathbf{m}$ be such that $aV = \mathbf{m}V$. Then $R[\mathbf{m}/a] \subset V$. Let $Q := \mathbf{m}(V) \cap R[\mathbf{m}/a]$. Then $R[\mathbf{m}/a]_Q$ is the **quadratic transformation of R along V** . In the special case where (R, \mathbf{m}) is a d -dimensional regular local domain we use the following terminology.

Definition 2.1. Let d be a positive integer and let (R, \mathbf{m}, k) be a d -dimensional regular local ring with maximal ideal \mathbf{m} and residue field k . Let $x \in \mathbf{m} \setminus \mathbf{m}^2$ and let $S_1 := R[\frac{\mathbf{m}}{x}]$. The ring S_1 is a d -dimensional regular ring in the sense that each localization of S_1 at a prime ideal is a regular local ring. To see this, observe that S_1/xS_1 is isomorphic to a polynomial ring in $d-1$ variables over the field k , cf. [SH, Corollary 5.5.9], and $S_1[1/x] = R[1/x]$ is a regular ring. Moreover, S_1 is a UFD since x is a prime element of S_1 and $S_1[1/x] = R[1/x]$ is a UFD, cf. [M, Theorem 20.2].

Let I is an \mathbf{m} -primary ideal of R with $r := \text{ord}_R(I)$. Then one has in S_1

$$IS_1 = x^r I_1 \quad \text{for some ideal } I_1 \text{ of } S_1.$$

We observe in Remark 2.2 that either $I_1 = S_1$ or $\text{ht } I_1 \geq 2$. Thus I_1 is the transform I^{S_1} of I in S_1 as in Definition 1.1.

Let \mathbf{p} be a prime ideal of $R[\frac{\mathbf{m}}{x}]$ with $\mathbf{m} \subseteq \mathbf{p}$. The local ring

$$R_1 := R[\frac{\mathbf{m}}{x}]_{\mathbf{p}} = (S_1)_{\mathbf{p}}$$

is called a **local quadratic transform** of R ; the ideal $I_1 R_1$ is the transform of I in R_1 as in Definition 1.1.

Remark 2.2. With the notation of Definition 2.1, to justify that the ideal I_1 is the transform of I in S_1 , we observe that the ideal I_1 is not contained in any height-one prime of S_1 . For if $I_1 \subseteq xS_1$, then we would have $I \subseteq x^{r+1}S_1 \cap R = \mathbf{m}^{r+1}$, a contradiction to the choice of r . If $I_1 \subseteq q$, where q is a height-one prime of S_1 different from xS_1 , then $I \subseteq q \cap R$. This is impossible since $q \cap R$ is a height-one prime of R and I is \mathbf{m} -primary.

We follow the notation of [L] and refer to regular local rings of dimension at least two as **points**. A point T is said to be **infinitely near** to a point R , in symbols, $R \prec T$, if there is a finite sequence of local quadratic transformations

$$(1) \quad R =: R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_n = T \quad (n \geq 0),$$

where R_{i+1} is a local quadratic transform of R_i for $i = 0, 1, \dots, n-1$. If such a sequence of local quadratic transforms as in Equation 1 exists, then it is unique and it is called the **quadratic sequence** from R to T [L, Definition 1.6].

Remark 2.3. Let (R, \mathbf{m}) be a regular local ring with $\dim R \geq 2$. As noted in [L, Proposition 1.7], there is a one-to-one correspondence between the points T infinitely near to R and the prime divisors V of R . This correspondence is defined by associating with T the order valuation ring V of T . Since V is the unique local quadratic transform of T of dimension one, the local quadratic sequence in Equation 1 extends to give Equation 2:

$$(2) \quad R =: R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_n = T \subset V.$$

The one-to-one correspondence between the points T infinitely near to R and the prime divisors V of R implies that T is the unique point infinitely near to R for which the order valuation ring of T is V . However, if $\dim R > 2$, then there often

exist regular local rings S with $S \neq T$ such that S birationally dominates R and the order valuation ring of S is V . We illustrate this in Example 2.4.

Example 2.4. Let (R, \mathbf{m}) be a 3-dimensional RLR with $\mathbf{m} = (x, y, z)R$, and let V denote the order valuation ring of R . Let $S = R[\frac{y}{x}]_{(x,z)}$. Then S is a 2-dimensional RLR that birationally dominates R , and V is the order valuation ring of S . Notice that S is not infinitely near to R .

Remark 2.5. Let (R, \mathbf{m}) be a d -dimensional RLR with $d \geq 2$ and let V be the order valuation ring of R . Let (S, \mathbf{n}) be a d -dimensional RLR that is a birational extension of R . Then

- (1) S dominates R .
- (2) If V dominates S , then $R = S$.
- (3) Thus R is the unique d -dimensional RLR having order valuation ring V among the regular local rings birational over R .

Proof. For item (1), let $P := \mathbf{n} \cap R$. Then $R_P \subseteq S$. If $P \neq \mathbf{m}$, then $\dim R_P = n < d$. Since every birational extension of an n -dimensional Noetherian domain has dimension at most n , we must have $\dim S \leq n$, a contradiction. Thus S dominates R . Item (2) follows from [Sa2, Corollary 2.6]. In more detail, if V dominates S , then $R/\mathbf{m} = S/\mathbf{n}$ and the elements in a minimal generating set for \mathbf{m} are part of a minimal generating set for \mathbf{n} . Hence we have $\mathbf{m}S = \mathbf{n}$. By Zariski's Main Theorem as in [N, (37.4)], it follows that $R = S$. Item (3) follows from item (2). \square

Example 2.6 demonstrates the existence of a prime divisor V for a 3-dimensional RLR (R, \mathbf{m}, k) for which there exist infinitely many distinct 3-dimensional RLRs that birationally dominate R , and have V as their order valuation ring.

Example 2.6. Let (R, \mathbf{m}, k) be a 3-dimensional regular local ring with residue field $R/\mathbf{m} = k$ and maximal ideal $\mathbf{m} = (x, y, z)R$. Let V be the prime divisor of R corresponding to the valuation v , where $v(x) = v(y) = 1$ and $v(z) = 3$, and where the images of $\frac{x^3}{z}$ and $\frac{y^3}{z}$ in the residue field k_v of V are algebraically independent over $R/\mathbf{m} = k$. Then we have :

- (1) In the unique finite sequence of local quadratic transformations given by [L, Proposition 1.7], we have:

$$R =: R_0 \subset R_1 \subset R_2 \subset V,$$

where

$$R_1 := R\left[\frac{\mathbf{m}}{x}\right]_{\mathbf{p}}, \quad \text{where } \mathbf{p} := \mathbf{m}(V) \cap R\left[\frac{\mathbf{m}}{x}\right] = \left(x, \frac{z}{x}\right)R\left[\frac{\mathbf{m}}{x}\right],$$

$$R_2 := R_1\left[\frac{\mathbf{m}_1}{x}\right]_{\mathbf{q}}, \quad \text{where } \mathbf{q} := \mathbf{m}(V) \cap R_1\left[\frac{\mathbf{m}_1}{x}\right] = \left(x, \frac{z}{x^2}\right)R_1\left[\frac{\mathbf{m}_1}{x}\right],$$

and V is the order valuation ring of R_2 . Notice that (R_1, \mathbf{m}_1) and (R_2, \mathbf{m}_2) are 2-dimensional RLRs.

- (2) For each integer $n \geq 1$, let $T_n := R\left[\frac{z}{x^2+y^{2n+1}}\right]_{(x,y,\frac{z}{x^2+y^{2n+1}})}$. Then we have:
- (a) By [Sa1, Lemma 4.2], each T_n is a 3-dimensional RLR that birationally dominates R .
 - (b) For each n , the images of $\frac{y}{x}$ and $\frac{z}{x^3}$ in k_v are algebraically independent over k . Hence the order valuation ring of T_n is V .
 - (c) By [Sa1, Corollary 4.5]), the elements in the family $\{T_n\}_{n=1}^\infty$ are distinct.

Definition 2.7. A **base point** of a nonzero ideal $I \subset R$ is a point T infinitely near to R such that $I^T \neq T$. The set of base points of I is denoted by

$$\mathcal{BP}(I) = \{ T \mid T \text{ is a point such that } R \prec T \text{ and } \text{ord}_T(I^T) \neq 0 \}.$$

The **point basis** of a nonzero ideal $I \subset R$ is the family of nonnegative integers

$$\mathcal{B}(I) = \{ \text{ord}_T(I^T) \mid R \prec T \}.$$

The nonzero ideal I is said to be **finitely supported** if I has only finitely many base points.

Definition 2.8. Let $R \prec T$ be points such that $\dim R = \dim T$. Lipman proves in [L, Proposition 2.1] the existence of a unique complete ideal P_{RT} in R such that for every point A with $R \prec A$, the complete transform

$$\overline{(P_{RT})^A} \text{ is } \begin{cases} \text{a *simple ideal if } A \prec T, \\ \text{the ring } A \text{ otherwise.} \end{cases}$$

The ideal P_{RT} of R is said to be a **special *-simple complete ideal**.

In the case where $R \prec T$ and $\dim R = \dim T$, we say that the order valuation ring of T is a **special prime divisor** of R .

Remark 2.9. With notation at in Definition 2.8, a prime divisor V of R is special if and only if the unique point T with $R \prec T$ such that the order valuation ring of T is V has $\dim T = \dim R$. Let $\dim R = d$. If V is a special prime divisor of R , then the residue field of V is a pure transcendental extension of degree $d - 1$ of the residue field $T/\mathbf{m}(T)$ of T , and $T/\mathbf{m}(T)$ is a finite algebraic extension of

R/\mathbf{m} . If the residue field R/\mathbf{m} of R is algebraically closed and V is a special prime divisor of R , then the residue field of V is a pure transcendental extension of R/\mathbf{m} of transcendence degree $d - 1$.

It would be interesting to identify and describe in other ways the special prime divisors of R among the set of all prime divisors of R .

3. FACTORIZATION AS PRODUCTS OF SPECIAL *-SIMPLE COMPLETE IDEALS

Let $R = \alpha$ be a d -dimensional regular local ring with $d \geq 2$. Lipman in [L, Theorem 2.5] proves that for every finitely supported complete ideal I of R there exists a unique family of integers

$$(n_\beta) = (n_\beta(I))_{\beta \succ \alpha, \dim \beta = \dim \alpha}$$

such that $n_\beta = 0$ for all but finitely many β and such that

$$(3) \quad \left(\prod_{n_\delta < 0} P_{\alpha\delta}^{-n_\delta} \right) * I = \prod_{n_\gamma > 0} P_{\alpha\gamma}^{n_\gamma}$$

where $P_{\alpha\beta}$ is the special *-simple ideal associated with $\alpha \prec \beta$ and the products are *-products. The product on the left in Equation 3 is over all $\delta \succ \alpha$ such that $n_\delta < 0$ and the product on the right is over all $\gamma \succ \alpha$ such that $n_\gamma > 0$.

Lipman gives the following example to illustrate this decomposition.

Example 3.1. Let k be a field and let $\alpha = R = k[[x, y, z]]$ be the formal power series ring in the 3 variables x, y, z over k . Let

$$\beta_x = R\left[\frac{y}{x}, \frac{z}{x}\right]_{(x, y/x, z/x)}, \quad \beta_y = R\left[\frac{x}{y}, \frac{z}{y}\right]_{(y, x/y, z/y)}, \quad \beta_z = R\left[\frac{x}{z}, \frac{y}{z}\right]_{(z, x/z, y/z)}$$

be the local quadratic transformations of R in the x, y, z directions. The associated special *-simple ideals are

$$P_{\alpha\beta_x} = (x^2, y, z)R, \quad P_{\alpha\beta_y} = (x, y^2, z)R, \quad P_{\alpha\beta_z} = (x, y, z^2)R.$$

The equation

$$(4) \quad (x, y, z)(x^3, y^3, z^3, xy, xz, yz) = P_{\alpha\beta_x} P_{\alpha\beta_y} P_{\alpha\beta_z}$$

represents the factorization of the finitely supported ideal $I = (x^3, y^3, z^3, xy, xz, yz)R$ as a product of special *-simple ideals. Here $P_{\alpha\alpha} = (x, y, z)R$. The base points of I are $\mathcal{BP}(I) = \{\alpha, \beta_x, \beta_y, \beta_z\}$ and the point basis of I is $\mathcal{B}(I) = \{2, 1, 1, 1\}$. Equation 4 represents the following equality of point bases

$$\mathcal{B}(P_{\alpha\alpha}) + \mathcal{B}(I) = \mathcal{B}(P_{\alpha\beta_x}) + \mathcal{B}(P_{\alpha\beta_y}) + \mathcal{B}(P_{\alpha\beta_z}).$$

Each of $P_{\alpha\beta_x}, P_{\alpha\beta_y}, P_{\alpha\beta_z}$ has a unique Rees valuation. Their product has in addition the order valuation of α as a Rees valuation.

Question 3.2. Let I be a finitely supported ideal of a regular local ring R .

- (1) If the base points of I are linearly ordered, does it follow that I is a $*$ -product of special $*$ -simple complete ideals, i.e., in the factorization given in Equation 3 are all the integers n_β nonnegative?
- (2) If I is $*$ -simple and if the base points of I are linearly ordered, does it follow that I is a special $*$ -simple ideal?
- (3) If $R \prec T$ with $\dim R = \dim T$ and $R \neq T$, can it happen that some power of the special $*$ -simple complete ideal P_{RT} has the maximal ideal \mathfrak{m} of R as a factor, that is, can there exist an ideal Q of R such that $\mathfrak{m}Q = (P_{RT})^n$ for some positive integer n ?¹

4. REES VALUATIONS OF IDEALS OF A REGULAR LOCAL RING

We use the following setting.

Setting 4.1. Let d be a positive integer and let (R, \mathfrak{m}, k) be a d -dimensional regular local ring with maximal ideal \mathfrak{m} and infinite residue field k . Let I be an \mathfrak{m} -primary ideal. For each $V \in \text{Rees } I$, let v denote the corresponding Rees valuation with value group \mathbb{Z} . Let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be such that $xV = \mathfrak{m}V$ for each $V \in \text{Rees } I$. Since the field k is infinite and the set $\text{Rees } I$ is finite, it is possible to choose such an element x . Let $r = \text{ord}_R I$. As in Definition 2.1, we have $IS_1 = x^r I_1$, where I_1 is the transform of I in $S_1 = R[\frac{\mathfrak{m}}{x}]$.

Remark 4.2. With the notation of Setting 4.1, we have:

- (1) If $J \subseteq I$ is a reduction of I in R , then $\text{ord}_R J = \text{ord}_R I = r$, and $JS_1 = x^r J_1$ is a reduction of $IS_1 = x^r I_1$ in S_1 . It follows that J_1 is the transform in S_1 of J and $J_1 \subseteq I_1$ is a reduction of I_1 in S_1 .
- (2) If $J = (a_1, \dots, a_d)R$ is a reduction of I then a DVR V that birationally dominates R is a Rees valuation ring of I if and only if the images of $\frac{a_2}{a_1}, \dots, \frac{a_d}{a_1}$ in the field $\frac{V}{\mathfrak{m}(V)}$ are algebraically independent over k .
- (3) The unique Rees valuation ring of \mathfrak{m} is $(S_1)_{xS_1}$, i.e., $\text{Rees } \mathfrak{m} = \{(S_1)_{xS_1}\}$.

¹In a joint paper with Matthew Toeniskoetter titled “Finitely supported $*$ -simple complete ideals in a regular local ring”, we have answered these three questions in the case where I is a finitely supported monomial ideal.

Proposition 4.3. *With the notation of Setting 4.1, we have:*

- (1) *If $I_1 = S_1$, then $v = \text{ord}_R$ and ord_R is the unique Rees valuation of I .*
- (2) *If $I_1 \neq S_1$ and $v \neq \text{ord}_R$, then $V \in \text{Rees}_{S_1} I_1$.*
- (3) *In general, we have $\text{Rees } I \subseteq \text{Rees}_{S_1} I_1 \cup \text{Rees } \mathbf{m}$.*

Proof. For the proof of item (1), let $\mathbf{p} := \mathbf{m}(V) \cap S_1$ be the center of V on S_1 . Since $xS_1 = \mathbf{m} S_1 \subseteq \mathbf{p}$ and $(S_1)_{xS_1}$ is the valuation ring of ord_R , it suffices to show that $\text{ht } \mathbf{p} = 1$. By the Dimension Formula ([M, page 119]), we have

$$\text{ht } \mathbf{p} = 1 \iff \text{tr.deg}_k \kappa(S_1/\mathbf{p}) = d - 1,$$

where $\kappa(S_1/\mathbf{p})$ denotes the field of fractions of S_1/\mathbf{p} . Let $J := (a_1, \dots, a_d)R$ be a reduction of I . Since $V \in \text{Rees } I = \text{Rees } J$, the images of $a_2/a_1, \dots, a_d/a_1$ in $V/\mathbf{m}(V) =: k_v$ are algebraically independent over k . Since JS_1 is a reduction of IS_1 and $IS_1 = x^r S_1$ is a principal ideal, we have

$$J_1 := (f_1, \dots, f_d)S_1 = S_1, \quad \text{where } f_i := \frac{a_i}{x^r} \text{ for } 1 \leq i \leq d.$$

It follows that $(f_1, \dots, f_d)V = V$ and thus $v(f_i) = 0$ for $i = 1, \dots, d$. Consider the inclusion maps:

$$\frac{S_1}{\mathbf{p}} \hookrightarrow \kappa(S_1/\mathbf{p}) \hookrightarrow \frac{V}{\mathbf{m}(V)}.$$

Since $v(f_i) = 0$ and $\mathbf{p} = \mathbf{m}(V) \cap S_1$, we have $f_i \in S_1 \setminus \mathbf{p}$. Therefore the images of $\frac{f_2}{f_1} = \frac{a_2}{a_1}, \dots, \frac{f_d}{f_1} = \frac{a_d}{a_1}$ in $\frac{V}{\mathbf{m}(V)}$ are in the subfield $\kappa(S_1/\mathbf{p})$ of $\frac{V}{\mathbf{m}(V)}$. Hence $\text{tr.deg}_k \kappa(S_1/\mathbf{p}) = d - 1$.

For the proof of item (2), we use the notation of the proof of item (1). Notice that f_1, \dots, f_d all have the same v -value. Moreover, since $V \neq (S_1)_{xS_1}$, we must have $v(f_i) > 0$; for if $v(f_i) = 0$, the proof of item (1) shows that $\text{ht } \mathbf{p} = 1$ and thus $V = (S_1)_{xS_1}$, a contradiction to our assumption. Thus $J_1 = (f_1, \dots, f_d)S_1 \subseteq \mathbf{p}$. Since J_1 is a reduction of I_1 and the images of $\frac{f_2}{f_1} = \frac{a_2}{a_1}, \dots, \frac{f_d}{f_1} = \frac{a_d}{a_1}$ in $\frac{V}{\mathbf{m}(V)}$ are algebraically independent over k , and thus we have $V \in \text{Rees}_{S_1} J_1 = \text{Rees}_{S_1} I_1$.

Item (3) follows from items (1) and (2). \square

Proposition 4.4. *Let the notation be as in Setting 4.1 and let $V \in \text{Rees } I$. As in Equation 2, there exists a unique finite sequence of local quadratic transforms*

$$(5) \quad R =: R_0 \subset R_1 \subset R_2 \subset \dots \subset R_n = T \subset V,$$

where V is the order valuation ring of T . Then the points R_0, \dots, R_n are all base points of I .

Proof. If V is the order valuation ring of R , then $n = 0$ in Equation 5 and R_0 is a base point of I . If $n > 0$, consider $S_1 = R[\frac{\mathfrak{m}}{x}]$ as in Setting 4.1. By Proposition 4.3, $V \in \text{Rees}_{S_1} I_1$. The local quadratic transform R_1 of R is a localization of S_1 and $I_1 R_1$ is a proper ideal in R_1 . By an inductive argument on the length n of the sequence in Equation 5, we conclude that the points R_1, \dots, R_n are base points of $I_1 R_1$ and therefore also base points of I . \square

Remark 4.5. Let the notation be as in Setting 4.1. If I is a finitely supported ideal in R , then [L, Corollary 1.22] implies that $\text{ht } I_1 = d$ and $\dim(S_1/I_1) = 0$.

Proposition 4.6. *Let the notation be as in Setting 4.1. If I is a finitely supported ideal in R , then*

$$\text{Rees}_{S_1} I_1 \subseteq \text{Rees } I.$$

Proof. By Proposition 4.3, we have $I_1 = S_1$ if and only if ord_R is the unique Rees valuation of I . Assume $I_1 \neq S_1$, and let $J := (a_1, \dots, a_d)R$ be a reduction of I . Since JS_1 is a reduction of IS_1 and $I_1 \neq S_1$, we have

$$J_1 := (f_1, \dots, f_d)S_1 \neq S_1, \quad \text{where} \quad f_i := \frac{a_i}{x^r} \quad \text{for} \quad 1 \leq i \leq d.$$

It follows that J_1 is a reduction of I_1 . By Remark 4.5, we have $\text{ht } J_1 = d$. Hence $\{f_1, \dots, f_d\}$ is a minimal set of generators of J_1 . For each $W \in \text{Rees}_{S_1} I_1 = \text{Rees}_{S_1} J_1$, the images of $f_2/f_1, \dots, f_d/f_1$ in $W/\mathfrak{m}(W)$ are algebraically independent over k . Let $\mathfrak{q} := \mathfrak{m}(W) \cap \overline{R[\frac{J}{a_1}]}$ be the center of W on $A := \overline{R[\frac{J}{a_1}]}$. Since $\frac{f_2}{f_1} = \frac{a_2}{a_1}, \dots, \frac{f_d}{f_1} = \frac{a_d}{a_1}$, we have that the images of $\frac{a_2}{a_1}, \dots, \frac{a_d}{a_1}$ in $\frac{W}{\mathfrak{m}(W)}$ are in the subfield $\kappa(A/\mathfrak{q})$ of $\frac{W}{\mathfrak{m}(W)}$. Hence $\text{tr.deg}_k \kappa(A/\mathfrak{q}) = d - 1$. By the Dimension Formula ([M, page 119]), $\text{ht}(\mathfrak{q}) = 1$. Hence $A_{\mathfrak{q}} = W$. \square

Propositions 4.3 and 4.6 imply the following corollary.

Corollary 4.7. *Let the notation be as in Setting 4.1. If I is a finitely supported ideal in R and ord_R is not a Rees valuation of I , then*

$$\text{Rees}_{S_1} I_1 = \text{Rees } I.$$

We use Proposition 4.8 to demonstrate that without the assumption in Proposition 4.6 that the ideal I has finite support, there sometimes exist Rees valuations of the transform I_1 of I that are not Rees valuations of I .

Proposition 4.8. *With the notation of Setting 4.1, if each Rees valuation of I is centered on a maximal ideal of S_1 and $\dim(S_1/I_1) > 0$, then there exist Rees valuations of I_1 that are not Rees valuations of I , i.e., $\text{Rees}_{S_1} I_1 \not\subseteq \text{Rees } I$.*

Proof. Since $\dim(S_1/I_1) > 0$, there exists a minimal prime P of I_1 such that P is not a maximal ideal of S_1 . Every minimal prime of I_1 is the center of at least one Rees valuation ring of I_1 . Let $V \in \text{Rees}_{S_1} I_1$ be centered on P . By assumption, $V \notin \text{Rees } I$. \square

We present in Example 4.9 a specific example where the hypotheses of Proposition 4.8 hold. By Proposition 4.6, the ideal I of Example 4.9 is not finitely supported.

Example 4.9. Let (R, \mathbf{m}, k) be a three-dimensional regular local ring with residue field $R/\mathbf{m} = k$ and maximal ideal $\mathbf{m} = (x, y, z)R$. Let

$$J := (x^2, y^3, z^5)R, \quad \text{and} \quad S_1 := R\left[\frac{\mathbf{m}}{z}\right] \quad \text{with} \quad x_1 := \frac{x}{z} \quad y_1 := \frac{y}{z}.$$

The ideal $I := (x^2, y^3, z^5, xy^2, xyz^2, y^2z^2, yz^4)R$ is the integral closure of J , and :

- (1) The ideals J and I have a unique Rees valuation v , where

$$v(x) = 15, \quad v(y) = 10, \quad \text{and} \quad v(z) = 6,$$

and the images of $\frac{x^2}{z^5}$ and $\frac{y^3}{z^5}$ in k_v are algebraically independent over k .

- (2) The center of v on S_1 is the maximal ideal $(x_1, y_1, z)S_1$.

- (3) $J_1 = (x_1^2, zy_1^3, z^3)S_1 \subset I_1 = (x_1^2, zy_1^3, z^3, x_1y_1^2z, x_1y_1z^2, y_1^2z^2)S_1$. We have J_1 is a reduction of I_1 with $\text{ht } J_1 = 2$ and $\mu(J_1) = 3$.

- (4) The ideal I is not finitely supported.

- (5) $\text{Rees}_{S_1} I_1 = \{V, W\}$, where V and W denote the valuation rings corresponding to v and w , respectively, and where

$$w(x) = 3, \quad w(y) = 2, \quad \text{and} \quad w(z) = 2,$$

and the images of $\frac{x^2}{y^3}$ and $\frac{x^2}{z^3}$ in k_w are algebraically independent over k .

- (6) $\text{Rees}_{S_1} I_1 \not\subseteq \text{Rees } I$.

Proof. The assertion in item (1) is well-known, see for example [SH, page 209], and item (2) follows from item (1). Since $I_1 \subseteq (x_1, z)S_1$, we have $\text{ht } I_1 = 2$ as asserted in item (3). Item (4) follows from Remark 4.5. For the proof of item (5), since v is not ord_R , Proposition 4.3 implies that $V \in \text{Rees}_{S_1} I_1$. We have $I_1 \subseteq \mathfrak{p} := (x_1, z)S_1$. Moreover :

- (1) y_1 is a unit in the two-dimensional regular local ring $(S_1)_{\mathfrak{p}}$. Also $\mathfrak{p} \cap R = \mathfrak{m}$ and the image of y_1 in the field of fractions of S_1/\mathfrak{p} is algebraically independent over R/\mathfrak{m} .
- (2) $(I_1)_{\mathfrak{p}} = (x_1^2, z)$ is a simple complete ideal in $(S_1)_{\mathfrak{p}}$.
- (3) $\text{Rees}(I_1)_{\mathfrak{p}} = \{W\}$, where $w(x_1) = 1$, $w(z) = 2$, and the image of $\frac{x_1^2}{z}$ in k_w is algebraically independent over the field of fractions of S_1/\mathfrak{p} .
- (4) $\text{Rees}_{S_1}(I_1) = \{V, W\}$, where $w(x) = 3$, $w(y) = z$, and $w(z) = 2$.

□

In Example 4.10, the height of the transform ideal I_1 is less than the height of I and yet $\text{Rees } I = \text{Rees}_{S_1} I_1$.

Example 4.10. Let (R, \mathfrak{m}, k) be a three-dimensional regular local ring with residue field $R/\mathfrak{m} = k$ and maximal ideal $\mathfrak{m} = (x, y, z)R$. Let

$$J := (x^2, y^2, z)R, \quad \text{and} \quad S_1 := R\left[\frac{\mathfrak{m}}{x}\right] \quad \text{with} \quad y_1 := \frac{y}{x} \quad z_1 := \frac{z}{x}.$$

The ideal $I := (x^2, xy, y^2, z)R$ is the integral closure of J , and :

- (1) The ideals J and I have a unique Rees valuation v , where

$$v(x) = 1, \quad v(y) = 1, \quad \text{and} \quad v(z) = 2,$$

and the images of $\frac{x^2}{z}$ and $\frac{y^2}{z}$ in k_v are algebraically independent over k .

- (2) $J_1 = I_1 = (x, z_1)S_1$, and hence $\text{ht } I_1 = 2$ and $\mu(I_1) = 2$.
- (3) I is not finitely supported.
- (4) $\text{Rees}_{S_1} I_1 = \text{Rees } I$.

Proof. Item (1) is well-known, cf. [SH, Theorem 10.3.5]. Item (2) is clear and item (3) follows from Remark 4.5. For the proof of item (4), since $\mathfrak{p} := I_1$ is a height two prime in S_1 . Then :

- (1) y_1 is unit in a two-dimensional regular local ring $(S_1)_{\mathfrak{p}}$. Also $\mathfrak{p} \cap R = \mathfrak{m}$ and the image of y_1 in the field of fractions of S_1/\mathfrak{p} is algebraically independent over R/\mathfrak{m} .
- (2) $\text{ord}_{(S_1)_{\mathfrak{p}}}$ is the unique Rees valuation of \mathfrak{p} . To see that this valuation is v , observe that $v(x) = v(z_1) = 1$ and the image of $\frac{z_1}{x}$ in k_v is algebraically independent over the subfield $(S_1)_{\mathfrak{p}}/\mathfrak{p}(S_1)_{\mathfrak{p}}$ of k_v . This follows because the images of y_1 and $\frac{z_1}{x}$ in k_v are algebraically independent over k .

□

5. PROJECTIVELY FULL FINITELY SUPPORTED COMPLETE IDEALS

We use the following definitions:

Definition 5.1. Let I be a regular proper ideal in a Noetherian ring R .

- (1) An ideal J in R is **projectively equivalent** to I , if some powers of I and J have the same integral closure, i.e., $\overline{I^j} = \overline{J^i}$ for some $i, j \in \mathbb{Z}^+$.
- (2) The ideal I is said to be **projective full**, if the only complete ideals that are projectively equivalent to I are the ideals $\overline{I^k}$ with $k \in \mathbb{Z}^+$.

The concept of projective equivalence of ideals was introduced by Samuel in [Sam] and further developed by Nagata in [Nag]. Making use of work of Rees in [Rees], McAdam, Ratliff, and Sally in [MRS, Corollary 2.4] prove that the set $\mathcal{P}(I)$ of complete ideals projectively equivalent to I is linearly ordered by inclusion and discrete. Moreover, if I and J are projectively equivalent, then $\text{Rees } I = \text{Rees } J$ and the values of I and J with respect to these Rees valuation rings are proportional [MRS, Proposition 2.10]. If there exists a projectively full ideal J that is projectively equivalent to I , then the set $\mathcal{P}(I)$ is said to be **projectively full**. As described in [CHRR1], there is naturally associated to the projective equivalence class of I a numerical semigroup $\mathcal{S}(I)$. One has $\mathcal{S}(I) = \mathbb{N}$, the semigroup of nonnegative integers under addition, if and only if $\mathcal{P}(I)$ is projectively full.

Remark 5.2. Let I be a finitely supported complete ideal of a d -dimensional regular local ring R , where $d \geq 2$.

- (1) Every ideal projectively equivalent to I is finitely supported.
- (2) If an ideal J is projectively equivalent to I , then the point bases $\mathcal{B}(I)$ and $\mathcal{B}(J)$ are proportional; indeed, if $\overline{I^n} = \overline{J^m}$ for positive integers n and m , then $n\mathcal{B}(I) = m\mathcal{B}(J)$. In particular, if I and J are projectively equivalent, then I and J have the same base points.
- (3) If the greatest common divisor (GCD) of the entries in $\mathcal{B}(I)$ is 1, then the ideal I is projectively full.
- (4) Every special $*$ -simple complete ideal is projectively full.

Proof. These statements all follow from [L, Remark 1.9 and Proposition 1.10]. We write out the details for item (3). Let $\mathcal{BP}(I) = \{R_0, R_1, \dots, R_s\}$, where $R_0 := R$ and $\mathcal{B}(I) = \{\text{ord}_{R_i}(I^{R_i})\}_{i=0}^s$. Let J be a complete ideal that is projectively equivalent to I . Then $\overline{I^n} = \overline{J^m}$ for some $n, m \in \mathbb{Z}^+$. By [L, Proposition 1.10], we have

$n\mathcal{B}(I) = \mathcal{B}(I^n) = \mathcal{B}(J^m) = m\mathcal{B}(J)$. Let $a_i := \text{ord}_{R_i}(I^{R_i})$ for $i = 0, \dots, s$. Then we have

$$n\mathcal{B}(I) = n\{a_0, a_1, \dots, a_s\} = m\{b_0, b_1, \dots, b_s\} = m\mathcal{B}(J),$$

where $b_i = \text{ord}_{R_i}(J^{R_i})$ for $i = 0, 1, \dots, s$. Since $\text{GCD}\{a_0, a_1, \dots, a_s\} = 1$, we have

$$n = n\left(\text{GCD}\{a_0, a_1, \dots, a_s\}\right) = \text{GCD}\{na_0, na_1, \dots, na_s\} = m\left(\text{GCD}\{b_0, b_1, \dots, b_s\}\right).$$

Hence $n = mr$, where $r = \text{GCD}\{b_0, b_1, \dots, b_s\}$, and therefore applying [L, Remark 1.9 and Proposition 1.10], we have

$$\begin{aligned} \overline{I^{mr}} = \overline{J^m} &\implies \mathcal{B}(I^{mr}) = \mathcal{B}(J^m) \\ &\iff mr\mathcal{B}(I) = m\mathcal{B}(J) \\ &\iff r\mathcal{B}(I) = \mathcal{B}(J) \\ &\iff \mathcal{B}(I^r) = \mathcal{B}(J) \\ &\iff \overline{I^r} = \overline{J} \\ &\iff \overline{I^r} = J, \quad \text{since } J \text{ is complete.} \end{aligned}$$

Thus I is projective full. Item (4) is immediate from item (3), because the last nonzero entry in the point basis of a special *-simple complete ideal is 1. \square

Remark 5.3. Let $(\alpha = R, \mathbf{m})$ be a d -dimensional regular local ring with $d \geq 2$, and let I be a complete finitely supported \mathbf{m} -primary ideal. Let $\mathcal{B}(I) = \{a_0, a_1, \dots, a_s\}$ be the point basis of I . By Lipman's unique factorization theorem: there exists a unique factorization as in Equation 3

$$\left(\prod_{n_\delta < 0} P_{\alpha^\delta}^{-n_\delta}\right) * I = \prod_{n_\gamma > 0} P_{\alpha^\gamma}^{n_\gamma}$$

If $d := \text{GCD}\{a_0, a_1, \dots, a_s\}$, the proof of this unique factorization implies that each exponent n_δ and n_γ is a multiple of d . In particular, if there are no negative exponents in this factorization, then there exists an ideal K such that $\overline{K^d} = I$.

In the two-dimensional case, the Zariski unique factorization theorem implies that $\mathcal{P}(I)$ is projectively full, and I is projectively full if and only if the GCD of the entries in the point basis of I is equal to 1.

In the higher dimensional case, we ask:

Question 5.4. Let I be a finitely supported complete ideal in a d -dimensional RLR.

- (1) If I is projectively full, does it follow that the GCD of the entries in the point basis of I is equal to 1?

- (2) If I is $*$ -simple, is I projectively full?
- (3) Is $\mathcal{P}(I)$ always projectively full?

Remark 5.5. With the notation as in Setting 4.1, it is possible that I is projectively full in R , while the transform I_1 is not projectively full in S_1 . For example, let $d = 2$ and $\mathbf{m} = (x, y)R$, and let

$$I = (x^2, y)^2 \mathbf{m} = (x^5, x^3y, xy^2, y^3)R.$$

Since \mathbf{m} is a simple factor of I , the ideal I is projectively full in R , cf. [CHRR2, Example 3.2]. We have $S_1 = R[\frac{y}{x}]$. Let $y_1 = \frac{y}{x}$. Then $IS_1 = x^3I_1$, where $I_1 = (x^2, xy_1, y_1^2)S_1$. Thus the ideal $I_1 = (x, y_1)^2S_1$ is not projectively full.

Proposition 5.6. *Let I be a complete \mathbf{m} -primary ideal of a regular local ring (R, \mathbf{m}) of dimension $d \geq 2$. With the notation as in Setting 4.1, if the transform I_1 of I in S_1 is projectively full, then I is projectively full in R .*

Proof. Let J be an ideal in R that is projectively equivalent to I , say $\overline{I^n} = \overline{J^m}$ with n, m positive integers. Assume that $r = \text{ord}_R I$ and $s = \text{ord}_R J$. Then $IS_1 = x^r I_1$ and $JS_1 = x^s J_1$. Thus taking complete transforms, we have

$$x^{rn} \overline{I_1^n} = \overline{I_1^n S_1} = \overline{J_1^m S_1} = x^{sm} \overline{J_1^m}.$$

Since neither of the ideals I_1 nor J_1 in the UFD S_1 is contained in a proper principal ideal of S_1 , we have $rn = sm$ and $\overline{I_1^n} = \overline{J_1^m}$. Thus I_1 and J_1 are projectively equivalent. Since I_1 is projectively full, $n = mt$ for some positive integer t . It follows that $\overline{I^t} = \overline{J}$. \square

6. THE STRUCTURE OF SPECIAL $*$ -SIMPLE COMPLETE IDEALS

Setting 6.1. We consider the structure of special $*$ -simple complete ideals as in Definition 2.8. In the case where $\dim R = 2$ and $R \prec T$, the special $*$ -simple complete ideal P_{RT} has a unique Rees valuation ord_T . In the higher dimensional case, the ideal P_{RT} has ord_T as a Rees valuation and often also has other Rees valuations. We observe in Proposition 6.4 that the other Rees valuations of P_{RT} are in the set $\{\text{ord}_{R_i}\}_{i=0}^{n-1}$, where

$$(6) \quad R =: R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_n = T \quad (n \geq 2),$$

where R_{i+1} is a local quadratic transform of R_i for $i = 0, 1, \dots, n-1$, and $\dim R = \dim T$. The residue field R_n/\mathbf{m}_n of R_n is a finite algebraic extension of the residue

field R_0/\mathfrak{m}_0 of R_0 . If $R_0/\mathfrak{m}_0 = R_n/\mathfrak{m}_n$, we observe in Corollary 6.6 that the other Rees valuations of P_{RT} are in the set $\{\text{ord}_{R_i}\}_{i=0}^{n-2}$.

Definition 6.2. We say **there is no change of direction** for the local quadratic sequence R_0 to R_n in Equation 6 if there exists an element $x \in \mathfrak{m}_0$ that is part of a minimal generating set of \mathfrak{m}_n . We say **there is a change of direction** between R_0 and R_n if $\mathfrak{m}_0 \subseteq \mathfrak{m}_n^2$.

Remark 6.3. With notation as in Setting 6.1, assume that $\dim R = \dim T$, and let $I = P_{R_0R_n}$.

- (1) By [L, Corollary 2.2], the transform $I^{R_j} = P_{R_jR_n}$ for all j with $0 \leq j \leq n$. By Proposition 4.6, we have $\text{Rees}_{R_j} I^{R_j} \subseteq \text{Rees } I$. Thus for each j with $0 \leq j \leq n$, we have

$$\text{Rees}_{R_j} P_{R_jR_n} = \text{Rees}_{R_j} I^{R_j} \subseteq \text{Rees } I,$$

and the number of Rees valuations of I is greater than or equal to the number of Rees valuations of $P_{R_jR_n}$.

- (2) If $R_0/\mathfrak{m}_0 = R_n/\mathfrak{m}_n$, then there is no change of direction in the local quadratic sequence from R_0 to $R_n \iff \text{ord}_{R_0}(I) = 1 \iff \mathcal{B}(I) = \{1, 1, \dots, 1, 1\}$.

Proposition 6.4. Let (R, \mathfrak{m}, k) be d -dimensional regular local ring, where $d \geq 2$, and let $R \prec T$ with $\dim T = d$. Assume the sequence of local quadratic transforms from R to T is as in Equation 6. Let $P_{R_0R_n}$ be the associated special $*$ -simple complete \mathfrak{m} -primary ideal in R , and let V_i denote the valuation ring of ord_{R_i} for $0 \leq i \leq n$. Then we have

$$\{V_n\} \subseteq \text{Rees } P_{R_0R_n} \subseteq \{V_0, V_1, \dots, V_{n-1}, V_n\}.$$

Proof. Let $I := P_{R_0R_n}$. Since $I^{R_n} = P_{R_nR_n}$ is the maximal ideal of R_n , we have $\{V_n\} \subseteq \text{Rees } P_{R_0R_n}$. Let $V \in \text{Rees } I$. We use the notation of Setting 4.1. Then $IS_1 = x^r I_1$, where $r := \text{ord}_R(I)$. By [L, Corollary (2.2)], I is a finitely supported ideal in R and

$$\mathcal{BP}(I) = \{R_0, R_1, \dots, R_n\}.$$

Hence R_1 is the only base point of I in the first neighborhood of R , and I_1 is contained in a unique maximal ideal N_1 in S_1 . Hence $R_1 = (S_1)_{N_1}$ and $\mathfrak{m}_1 := N_1 R_1$.

By Proposition 4.3, we have

$$\begin{aligned} \operatorname{Rees} P_{R_0 R_n} &= \operatorname{Rees} I \subseteq \operatorname{Rees}_{S_1} I_1 \cup \operatorname{Rees} \mathbf{m} \\ &= \operatorname{Rees}_{R_1} I_1 R_1 \cup \operatorname{Rees} \mathbf{m} \\ &= \operatorname{Rees}_{R_1} P_{R_1 R_n} \cup \operatorname{Rees} \mathbf{m}. \end{aligned}$$

Let \mathbf{m}_i denote the maximal ideal of R_i for $i = 1, \dots, n$. Since $I^{R_1} = P_{R_1 R_n}$, a simple induction argument proves that

$$\begin{aligned} \operatorname{Rees} P_{R_0 R_n} &\subseteq \operatorname{Rees}_{R_1} P_{R_1 R_n} \cup \operatorname{Rees} \mathbf{m} \\ &\subseteq \operatorname{Rees}_{R_2} P_{R_2 R_n} \cup \operatorname{Rees}_{R_1} \mathbf{m}_1 \cup \operatorname{Rees} \mathbf{m} \\ &\subseteq \dots \\ &\subseteq \operatorname{Rees}_{R_n} \mathbf{m}_n \cup \operatorname{Rees}_{R_{n-1}} \mathbf{m}_{n-1} \cup \operatorname{Rees}_{R_{n-2}} \mathbf{m}_{n-2} \cup \dots \cup \operatorname{Rees}_{R_1} \mathbf{m}_1 \cup \operatorname{Rees} \mathbf{m}. \end{aligned}$$

□

We describe in Remark 6.5 the structure of a special $*$ -simple complete ideal $P_{R_0 R_1}$ in the case where $R_1/\mathbf{m}_1 = R_0/\mathbf{m}_0$. This case always occurs if R_0/\mathbf{m}_0 is algebraically closed.

Remark 6.5. Let $R = R_0$ be a d -dimensional regular local ring and let R_1 be a local quadratic transform of R with $\dim R_1 = d$. Let $P_{R_0 R_1}$ be the associated special $*$ -simple complete ideal of R_0 . With notation as in Setting 4.1, we may assume that

$$S_1 = R_0 \left[\frac{\mathbf{m}_0}{x_1} \right] = R_0 \left[\frac{x_2}{x_1}, \dots, \frac{x_d}{x_1} \right] \subset R_1,$$

where $\mathbf{m}_0 := \mathbf{m} = (x_1, \dots, x_d)R_0$. Then $R_1 = (S_1)_{N_1}$, where N_1 is a maximal ideal in S_1 containing $x_1 S_1 = \mathbf{m} S_1$. Assume that $R_1/\mathbf{m}_1 = R_0/\mathbf{m}_0$.

- (1) Then $N_1 = (x_1, \frac{x_2}{x_1} - a_2, \dots, \frac{x_d}{x_1} - a_d)S_1$, where $a_2, \dots, a_d \in R_0$. We have

$$P_{R_0 R_1} = (x_1^2, x_2 - a_2 x_1, \dots, x_d - a_d x_1)R_0,$$

and the ideal $P_{R_0 R_1}$ has unique Rees valuation $w := \operatorname{ord}_{R_1}$, where

$$w(x_1) = 1 \quad \text{and} \quad w(x_i - a_i x_1) = 2 \quad \text{for} \quad i = 2, \dots, d,$$

and the images of $\frac{x_2 - a_2 x_1}{x_1^2}, \dots, \frac{x_d - a_d x_1}{x_1^2}$ in k_w are algebraically independent over R_0/\mathbf{m}_0 . Thus $\operatorname{Rees} P_{R_0 R_1} = \operatorname{Rees}_{R_1} \mathbf{m}_1$.

- (2) $\mathcal{BP}(P_{R_0 R_1}) = \{R_0, R_1\}$.

- (3) $\mathcal{B}(P_{R_0 R_1}) = \{1, 1\}$.

- (4) The ideal $I := P_{R_0 R_1}$ is a normal ideal cf. [Go]. Hence the Rees algebra $R[It]$ is a normal domain. Also $\frac{R[It]}{Q} \cong (\frac{R}{\mathbf{m}})[T_1, \dots, T_d]$ is a polynomial ring in d -variables over R/\mathbf{m} , where $\operatorname{Min}(\mathbf{m} R[It]) = \{Q\}$ and $Q = \mathbf{m} R[It]$.

As a consequence of Proposition 6.4 and Remark 6.5, we have

Corollary 6.6. *Let the notation be as in Proposition 6.4. Assume that $R_0/\mathbf{m}_0 = R_n/\mathbf{m}_n$. Then we have*

$$\{V_n\} \subseteq \text{Rees } P_{R_0 R_n} \subseteq \{V_0, V_1, \dots, V_{n-2}, V_n\}.$$

With notation as in Setting 6.1, we illustrate in Example 6.7 the structure of the special *-simple complete ideal $I = P_{R_0 R_n}$ in the case where $R_0/\mathbf{m}_0 = R_n/\mathbf{m}_n$ and there is no change of direction. We assume $\dim R_0 = 3$. The situation is similar for $\dim R_0 > 3$.

Example 6.7. Let (R, \mathbf{m}_0, k) be a 3-dimensional regular local ring with maximal ideal $\mathbf{m}_0 = (x, y, z)R$, and let a_1, \dots, a_n and b_1, \dots, b_n be elements in R . Consider the following finite sequence of local quadratic transformations

$$R =: R_0 \subset {}^x R_1 \subset {}^{xx} R_2 \subset {}^{xxx} R_3 \subset \dots \subset \overbrace{{}^{x \cdots x} R_n}^{n \text{ times}},$$

where for $i = 0, 1, 2, \dots, n-1$ we define S_{i+1} and $\overbrace{{}^{x \cdots x} R_{i+1}}^{i+1 \text{ times}} = R_{i+1}$ inductively by

$$\begin{aligned} S_1 &:= R_0\left[\frac{\mathbf{m}_0}{x}\right], \quad N_1 := \left(x, \frac{y - a_1 x}{x}, \frac{z - b_1 x}{x}\right)S_1 \quad R_1 := (S_1)_{N_1} \quad \mathbf{m}_1 := N_1 R_1 \\ &\dots \\ S_{i+1} &:= R_i\left[\frac{\mathbf{m}_i}{x}\right], \quad N_{i+1} := \left(x, \frac{y - a_1 x - \dots - a_{i+1} x^{i+1}}{x^{i+1}}, \frac{z - b_1 x - \dots - b_{i+1} x^{i+1}}{x^{i+1}}\right)S_{i+1} \\ R_{i+1} &:= (S_{i+1})_{N_{i+1}} \quad \mathbf{m}_{i+1} := N_{i+1} R_{i+1}. \end{aligned}$$

Then for $i = 0, 1, \dots, n$, we have

- (1) The order valuation $v_i := \text{ord}_{R_i}$ has values $v_i(x) = 1$ and

$$v_i\left(\frac{y - a_1 x - \dots - a_i x^i}{x^i}\right) = v_i\left(\frac{z - b_1 x - \dots - b_i x^i}{x^i}\right) = 1.$$

and the images of

$$\frac{y - a_1 x - \dots - a_i x^i}{x^{i+1}} \quad \text{and} \quad \frac{z - b_1 x - \dots - b_i x^i}{x^{i+1}}$$

in the residue field k_{v_i} of V_i are algebraically independent over R_0/\mathbf{m}_0 .

- (2) The special *-simple complete \mathbf{m} -primary ideal is

$$P_{R_0 R_i} = (x^{i+1}, y - a_1 x - \dots - a_i x^i, z - b_1 x - \dots - b_i x^i)R.$$

- (3) $\mathcal{BP}(P_{R_0 R_i}) = \{R_0, R_1, R_2, \dots, R_i\}$.

- (4) $\mathcal{B}(P_{R_0 R_i}) = \{1, 1, 1, \dots, 1\}$.

- (5) The special $*$ -simple complete \mathbf{m} -primary ideal $P_{R_0 R_i}$ has a unique Rees valuation ord_{R_i} . That is, $\text{Rees}(P_{R_0 R_i}) = \text{Rees}_{R_i} \mathbf{m}_i$.
- (6) The ideal $P_{R_0 R_i}$ is normal.
- (7) Let $I := P_{R_0 R_i}$. Then $\mathbf{m} R[It]$ has a unique minimal prime $Q := \mathbf{m} R[It]$ and $\frac{R[It]}{Q}$ is a polynomial ring in 3-variables over R/\mathbf{m} .

Theorem 6.8. *Let the notation be as in Setting 6.1. Assume that $R_0/\mathbf{m}_0 = R_n/\mathbf{m}_n$. Then the following are equivalent:*

- (1) $\text{Rees } P_{R_0 R_n} = \text{Rees}_{R_n} \mathbf{m}_n$, i.e., ord_{R_n} is the unique Rees valuation of $P_{R_0 R_n}$.
- (2) Either $\dim R_0 = 2$, or there is no change of direction in the local quadratic sequence given in Equation 6.

Proof. (2) \implies (1): If $\dim R_0 = 2$, then the theory of Zariski implies that I has a unique Rees valuation. Assume that $\dim R_0 \geq 3$ and that there is no change of direction in the sequence given in Equation 6. By Example 6.7, we have that the special $*$ -simple complete ideal $P_{R_0 R_n}$ has the unique Rees valuation, ord_{R_n} .

(1) \implies (2): First, notice that $\text{Rees}_{R_n} \mathbf{m}_n \subseteq \text{Rees } P_{R_0 R_n}$ by Proposition 6.4, and hence $|\text{Rees } P_{R_0 R_n}| \geq 1$. To conclude the proof, we prove the following :

Claim 6.9. *If there is at least one change of direction in the local quadratic sequence given in Equation 6, then $|\text{Rees } P_{R_0 R_n}| > 1$.*

Proof. Assume there is at least one change of direction between R_0 and R_n . Choose j minimal so that there is no change of direction from R_{j+1} to R_n . Then by choosing appropriate regular parameters x in R_j and y in R_{j+1} , we have the following local quadratic sequence:

$$R =: R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_j \subset R_{j+1} \subset R_{j+2} \subset R_{j+3} \subset \cdots \subset R_n \quad .$$

$\begin{array}{ccccccc} \overset{A_0}{\parallel} & \overset{x A_1}{\parallel} & \overset{yx A_2}{\parallel} & \overset{yyx A_3}{\parallel} & & \overset{\overset{n-j \text{ times}}{\parallel}}{y \cdots yx} & \overset{A_{n-j}}{\parallel} \end{array}$

By Remark 6.3, we have $\text{Rees}_{R_j} P_{R_j R_n} \subseteq \text{Rees } P_{R_0 R_n}$. Thus to complete the proof of the Claim, we analyse in Example 6.10 the structure of a special $*$ -simple complete \mathbf{m} -primary ideal of a $d \geq 3$ -dimensional regular local ring obtained by a change of direction first dividing by x and then successively by y . For notational simplicity, we assume that $d = 3$. The pattern is similar in the case where $d > 3$. \square

Example 6.10. Let (R, \mathbf{m}, k) be a 3-dimensional regular local ring with maximal ideal $\mathbf{m} = (x, y, z)R$. Let $n \geq 3$. Consider a sequence of local quadratic transforms

$$R := R_0 \subset R_1 := {}^x R_1 \subset R_2 := {}^{yx} R_2 \subset R_3 := {}^{yyx} R_3 \subset \cdots \subset R_n := \overbrace{y \cdots yyx}^{n \text{ times}} R_n$$

defined by

$$\begin{aligned} S_1 &:= R[\frac{\mathbf{m}}{x}], & N_1 &:= (x, \frac{y}{x}, \frac{z}{x})S_1, & R_1 &:= (S_1)_{N_1}, \mathbf{m}_1 := N_1 R_1 \\ S_2 &:= R_1[\frac{\mathbf{m}_1}{y/x}], & N_2 &:= (\frac{x^2}{y}, \frac{y}{x}, \frac{z - b_2 y}{y})S_2, & R_2 &:= (S_2)_{N_2}, \mathbf{m}_2 := N_2 R_2 \\ S_3 &:= R_2[\frac{\mathbf{m}_2}{y/x}], & N_3 &:= (\frac{x^3 - a_3 y^2}{y^2}, \frac{y}{x}, \frac{xz - b_2 xy - b_3 y^2}{y^2})S_3, & R_3 &:= (S_3)_{N_3}, \mathbf{m}_3 := N_3 R_3 \\ &\dots & & & & \\ S_n &:= R_{n-1}[\frac{\mathbf{m}_{n-1}}{y/x}], & N_n &:= (f, g, h)S_n, & R_n &:= (S_n)_{N_n}, \mathbf{m}_n := N_n R_n, \end{aligned}$$

where

$$\begin{aligned} f &:= \frac{x^n - a_3 x^{n-3} y^2 - \cdots - a_{n-1} x y^{n-2} - a_n y^{n-1}}{y^{n-1}} \\ g &:= \frac{y}{x} \\ h &:= \frac{x^{n-2} z - b_2 x^{n-2} y - \cdots - b_{n-1} x y^{n-2} - b_n y^{n-1}}{y^{n-1}}. \end{aligned}$$

Here the elements a_i and b_j are in R_0 , and we are assuming that $R_0/\mathbf{m}_0 = R_n/\mathbf{m}_n$. Thus we may choose x, y, z so that $N_1 = (x, \frac{y}{x}, \frac{z}{x})S_1$. We are also assuming that there is a change of direction from R_0 to R_2 . Thus we may assume $N_2 = (\frac{x^2}{y}, \frac{y}{x}, \frac{z - b_2 y}{y})S_2$.

Let

$$\begin{aligned} f_0 &:= x^n - a_3 x^{n-3} y^2 - \cdots - a_{n-1} x y^{n-2} - a_n y^{n-1} \\ h_0 &:= x^{n-2} z - b_2 x^{n-2} y - \cdots - b_{n-1} x y^{n-2} - b_n y^{n-1}. \end{aligned}$$

Then:

(1) Let $v_n := \text{ord}_{R_n}$. We have

$$v_n(f_0) = 1 + (n-1)v_n(y),$$

$$(7) \quad v_n(y) = 1 + v_n(x),$$

$$v_n(h_0) = 1 + (n-1)v_n(y).$$

(2) Let

$$K := \{\alpha \in \mathbf{m}_0 \mid v_n(\alpha) \geq v_n(y^n) \text{ and } v_0(\alpha) \geq n\}.$$

Then we have $K = P_{R_0 R_n}$

$$(3) \quad \mathcal{BP}(P_{R_0 R_n}) = \{R_0, R_1, R_2, R_3, \dots, R_n\}.$$

$$(4) \quad \mathcal{B}(P_{R_0 R_n}) = \{n, 1, 1, \dots, 1, 1\}.$$

(5) The Rees valuations of $P_{R_0 R_n}$ are ord_{R_0} and ord_{R_n} .

Proof. For item (1), since $\mathbf{m}_n = (f, g, h)R_n$ we have

$$\begin{aligned} 1 &= v_n\left(\frac{x^n - a_3x^{n-3}y^2 - \cdots - a_{n-1}xy^{n-2} - a_ny^{n-1}}{y^{n-1}}\right) = v_n\left(\frac{y}{x}\right) \\ &= v_n\left(\frac{x^{n-2}z - b_2x^{n-2}y - \cdots - b_{n-1}xy^{n-2} - b_ny^{n-1}}{y^{n-1}}\right), \end{aligned}$$

and hence

$$v_n(f_0) = 1 + (n-1)v_n(y), \quad v_n(y) = 1 + v_n(x), \quad v_n(h_0) = 1 + (n-1)v_n(y).$$

Multiplying the listed generators of \mathbf{m}_n by xy^{n-1} , we obtain elements in R_0

$$x \cdot f_0, \quad y^n, \quad x \cdot h_0$$

By Equation 7, we have

$$n(1 + v_n(x)) = v_n(xf_0) = v_n(y^n) = v_n(xh_0).$$

For item (2), we clearly have $K \supseteq (xf_0, y^n, xh_0)R_0$.

We observe that $v_n(z) \geq v_n(y)$; for if $v_n(z) < v_n(y)$, then

$$v_n(x^{n-1}z) < v_n(-b_2x^{n-1}y - \cdots - b_nxy^{n-1})$$

implies that $v_n(x^{n-1}z) = v_n(xh_0) = v_n(y^n)$, a contradiction. Therefore $z^n \in K$.

Since K has order n and contains y^n and z^n , we see that the transform of K in $R_0[\frac{\mathbf{m}_0}{y}]$ and in $R_0[\frac{\mathbf{m}_0}{z}]$ is the whole ring. Let $y_1 := \frac{y}{x}$ and $z_1 := \frac{z}{x}$, then $y = xy_1$ and $z = xz_1$. Hence

$$\begin{aligned} KR_0\left[\frac{\mathbf{m}_0}{x}\right] &= KS_1 \\ &\supseteq x^n \left(x - (a_3y_1^2 - \cdots - a_ny_1^{n-1}), y_1^n, z_1 - (b_2y_1 - b_3y_1^2 - \cdots - b_ny_1^{n-1}) \right) S_1. \end{aligned}$$

Thus the transform of K in S_1 is

$$K^{S_1} \supseteq \left(x - (a_3y_1^2 - \cdots - a_ny_1^{n-1}), y_1^n, z_1 - (b_2y_1 - b_3y_1^2 - \cdots - b_ny_1^{n-1}) \right) S_1.$$

Since the ideal K^{S_1} is primary for the maximal ideal $N_1 = (x_1, y_1, z_1)S_1$ and $R_1 = (S_1)_{N_1}$, we have

$$K^{S_1} = \left(x - (a_3y_1^2 - \cdots - a_ny_1^{n-1}), y_1^n, z_1 - (b_2y_1 - b_3y_1^2 - \cdots - b_ny_1^{n-1}) \right) S_1,$$

and hence

$$K^{R_1} = \left(x - (a_3y_1^2 - \cdots - a_ny_1^{n-1}), y_1^n, z_1 - (b_2y_1 - b_3y_1^2 - \cdots - b_ny_1^{n-1}) \right) R_1.$$

As in Example 6.7, we have

$$P_{R_1 R_n} = \left(x - (a_3y_1^2 - \cdots - a_ny_1^{n-1}), y_1^n, z_1 - (b_2y_1 - b_3y_1^2 - \cdots - b_ny_1^{n-1}) \right) R_1,$$

and thus $K^{R_1} = P_{R_1 R_n}$. By [L, Proposition 2.1], we have $K = P_{R_0 R_n}$. Items (3) and (4) are clear. Since K has order n and contains y^n, z^n and xh_0 , we see that ord_{R_0} is a Rees valuation of K . Therefore ord_{R_0} and ord_{R_n} are the Rees valuations of K . \square

This completes the proof of Theorem 6.8. \square

We illustrate in Examples 6.11 and 6.12 the behavior of a special $*$ -simple complete ideal $P_{R_0 R_1}$ in cases where $[R_1/\mathbf{m}_1 : R_0/\mathbf{m}_0] > 1$. In Example 6.11 the ideal $P_{R_0 R_1}$ has two Rees valuations, while in Example 6.12 the ideal $P_{R_0 R_1}$ has only one Rees valuation. We use notation as in Remark 6.5 with $d = 3$ and $\mathbf{m}_0 = (x, y, z)R_0$.

Example 6.11. Let $R_0/\mathbf{m}_0 = \mathbb{Q}$ and $R_1 := (S_1)_{N_1}$, where

$$N_1 := (x, \frac{y}{x}, (\frac{z}{x})^2 - 3)S_1.$$

Let $w := \text{ord}_{R_1}$. Then we have

(1)

$$w(x) = w(\frac{y}{x}) = w((\frac{z}{x})^2 - 3) = 1,$$

and the images of $\frac{y}{x^2}, \frac{z^2 - 3x^2}{x^3}$ in the residue field k_w of w are algebraically independent over R_0/\mathbf{m}_0 . Also $w(z^2 - 3x^2) = 1 + w(x^2) = 3$. Therefore

x	y	z	$z^2 - 3x^2$
$w := \text{ord}_{R_1}$	1	2	1
			3

(2) Let

$$I := \{\alpha \in \mathbf{m}_0 \mid w(\alpha) \geq 3\}.$$

Then we have

(a) $I = (x^3, xy, z^2 - 3x^2, y^2, yz, z^3)R_0$. A direct computation shows that

$I = P_{R_0 R_1}$. We have

$P_{R_0 R_1}$	x^3	xy	$z^2 - 3x^2$	y^2	yz	z^3
$w := \text{ord}_{R_1}$	3	3	3	4	3	3
$v := \text{ord}_{R_0}$	3	2	2	2	2	3

(b) $\mathcal{BP}(P_{R_0 R_1}) = \{R_0, R_1\}$.

(c) $\mathcal{B}(P_{R_0 R_1}) = \{2, 1\}$.

(d) The set of Rees valuations of $P_{R_0 R_1}$ is $\{\text{ord}_{R_0}, \text{ord}_{R_1}\}$.

Example 6.12. Let $R_0/\mathbf{m}_0 = \mathbb{Q}$ and $R_1 := (S_1)_{N_1}$, where

$$N_1 := (x, (\frac{y}{x})^2 - 2, (\frac{z}{x})^2 - 3)S_1.$$

Let $w := \text{ord}_{R_1}$. Then we have

(1)

$$w(x) = w\left(\left(\frac{y}{x}\right)^2 - 2\right) = w\left(\left(\frac{z}{x}\right)^2 - 3\right) = 1,$$

and the images of $\frac{y^2-2x^2}{x^3}, \frac{z^2-3x^2}{x^3}$ in the residue field k_w of w are algebraically independent over R_0/\mathbf{m}_0 . Also $w(y^2 - 2x^2) = w(z^2 - 3x^2) = 1 + w(x^2) = 3$.

Therefore

$$\frac{\quad}{w := \text{ord}_{R_1}} \left| \begin{array}{c|c|c} x & y & z \\ \hline 1 & 1 & 1 \end{array} \right|$$

(2) Let

$$I := \{\alpha \in \mathbf{m}_0 \mid w(\alpha) \geq 3\}.$$

Then we have

- (a) $I = (y^2 - 2x^2, z^2 - 3x^2, \mathbf{m}_0^3)R_0$. A direct computation shows that $I = P_{R_0R_1}$. We have

$P_{R_0R_1}$	$y^2 - 2x^2$	$z^2 - 3x^2$	\mathbf{m}_0^3
$w := \text{ord}_{R_1}$	3	3	3
$v := \text{ord}_{R_0}$	2	2	3

- (b) $\mathcal{BP}(P_{R_0R_1}) = \{R_0, R_1\}$.

- (c) $\mathcal{B}(P_{R_0R_1}) = \{2, 1\}$.

- (d) $\text{Rees}(P_{R_0R_1}) = \text{Rees}_{R_1} \mathbf{m}_1$.

Example 6.13 illustrates a pattern with exactly one change of direction from R_0 to R_2 where $R_0/\mathbf{m}_0 = R_2/\mathbf{m}_2$.

Example 6.13. Let (R, \mathbf{m}, k) be a 3-dimensional regular local ring with maximal ideal $\mathbf{m} = (x, y, z)R$. Consider the following sequence of local quadratic transforms

$$R := R_0 \subset R_1 := {}^x R_1 \subset R_2 := {}^{yx} R_2$$

defined by

$$\begin{aligned} S_1 &:= R\left[\frac{\mathbf{m}}{x}\right], & N_1 &:= \left(x, \frac{y}{x}, \frac{z}{x}\right)S_1, & R_1 &:= (S_1)_{N_1}, & \mathbf{m}_1 &:= N_1 R_1 \\ S_2 &:= R_1\left[\frac{\mathbf{m}_1}{y/x}\right], & N_2 &:= \left(\frac{x^2}{y}, \frac{y}{x}, \frac{z - b_2 y}{y}\right)S_2, & R_2 &:= (S_2)_{N_2}, & \mathbf{m}_2 &:= N_2 R_2 \end{aligned}$$

Then:

- (1) Let $v_2 := \text{ord}_{R_2}$. Then $v_2(x) = 2$, $v_2(y) = 3$, $v_2(z - b_2 y) = 4$, and the images of $\frac{x^3}{y^2}, \frac{x(z - b_2 y)}{y^2}$ in the residue field k_{v_2} of V_2 are algebraically independent over $R_2/\mathbf{m}_2 = k$.

(2) The special $*$ -simple complete \mathbf{m} -primary ideal $P_{R_0R_2}$ is a v_2 -ideal. We have

$$\begin{aligned} P_{R_0R_2} &= \{\alpha \in \mathbf{m} \mid v_2(\alpha) \geq 6\} \\ &= (x^3, x(z - b_2y), y^2, x^2y, y(z - b_2y), (z - b_2y)^2)R. \end{aligned}$$

(3) $\mathcal{BP}(P_{R_0R_2}) = \{R_0, R_1, R_2\}$.

(4) $\mathcal{B}(P_{R_0R_2}) = \{2, 1, 1\}$.

(5) The set of Rees valuations of $P_{R_0R_2}$ is $\{\text{ord}_{R_0}, \text{ord}_{R_2}\}$.

(6) Let $I := P_{R_0R_2}$. Then :

(a) $\text{Min}(\mathbf{m}R[It]) = \{P_0, P_2\}$, where

$$\begin{aligned} P_2 &= (\mathbf{m}, x^2yt, y(z - b_2y)t, (z - b_2y)^2t)R[It] \quad \text{and} \\ P_0 &= (\mathbf{m}, x^3t, x^2yt)R[It]. \end{aligned}$$

(b) Let V_0 and V_2 denote the valuation rings corresponding to $v_0 := \text{ord}_{R_0}$ and $v_2 := \text{ord}_{R_2}$, where $V_0 = R[It]_{P_0} \cap \mathcal{Q}(R)$ and $V_2 = R[It]_{P_2} \cap \mathcal{Q}(R)$. Then $\frac{R[It]}{P_2}$ is a polynomial ring in 3-variables over R/\mathbf{m} , and $\frac{R[It]}{P_0} \cong \frac{(R/\mathbf{m})[T_1, T_2, T_3, T_4]}{(T_2T_4 - T_3^2)}$ is a 3-dimensional normal Cohen-Macaulay domain with minimal multiplicity at its maximal homogeneous ideal with this multiplicity being 2.

Proof. (1) : Since $\mathbf{m}_2 = (\frac{x^2}{y}, \frac{y}{x}, \frac{z-b_2y}{y})R_2$, we have $v_2(\frac{x^2}{y}) = v_2(\frac{y}{x}) = v_2(\frac{z-b_2y}{y}) = 1$, and hence $v_2(x) = 2, v_2(y) = 3, v_2(z - b_2y) = 4$, and the images of $\frac{x^3}{y^2}, \frac{x(z-b_2y)}{y^2}$ in the residue field k_{v_2} of V_2 are algebraically independent over $R_2/\mathbf{m}_2 = k$.

(2), (3), and (4) : The transform in R_2 of the ideal $K := (x^3, y^2, x(z - b_2y))R$ is the maximal ideal $\mathbf{m}_2 = (\frac{x^2}{y}, \frac{y}{x}, \frac{z-b_2y}{y})R_2$ and $v_2(K) = 6$. Let

$$I := \{\alpha \in \mathbf{m} \mid v_2(\alpha) \geq 6\} = (x^3, x(z - b_2y), y^2, x^2y, y(z - b_2y), (z - b_2y)^2)R.$$

The ideal I is a complete \mathbf{m} -primary ideal. We see by direct computation that:

(a) $I^{R_1} = (x, (\frac{y}{x})^2, \frac{z-b_2y}{x})R_1$ is a special $*$ -simple complete ideal in R_1 .

(b) $I^{R_2} = \mathbf{m}_2$.

(c) $\mathcal{BP}(I) = \{R_0, R_1, R_2\}$.

Thus by [L, Proposition 2.1], we have $I = P_{R_0R_2}$. It is clear that $\text{ord}_{R_0}(I) = 2, \text{ord}_{R_1}(I^{R_1}) = 1, \text{ord}_{R_2}(I^{R_2}) = 1$. Hence $\mathcal{B}(I) = \{2, 1, 1\}$.

(5) : By item (1), $v_2 := \text{ord}_{R_2}$ is a Rees valuation of $P_{R_0R_2}$. We have the following table :

$P_{R_0R_2}$	x^3	$x(z - b_2y)$	y^2	x^2y	$y(z - b_2y)$	$(z - b_2y)^2$
$v_2 := \text{ord}_{R_2}$	6	6	6	7	7	8
$v_1 := \text{ord}_{R_1}$	3	3	4	4	4	4
$v_0 := \text{ord}_{R_0}$	3	2	2	3	2	2

Since the images of $\frac{y^2}{yz}, \frac{x(z-b_2y)}{yz}$ in the residue field k_{v_0} of V_0 are algebraically independent over k , $v_0 := \text{ord}_{R_0}$ is a Rees valuation of $P_{R_0 R_2}$. Using Proposition 6.4, we conclude that $P_{R_0 R_2}$ has the two Rees valuations, $v_0 := \text{ord}_{R_0}$ and $v_2 := \text{ord}_{R_2}$.

(6) : The statements of item (6) follow from the previous items and the connection between the Rees valuations of I and minimal primes of $\mathbf{m} R[It]$ in the Rees algebra $R[It]$, cf. [HK]. \square

Example 6.14 illustrates a pattern where there are exactly two changes of direction from R_0 to R_3 and where $R_0/\mathbf{m}_0 = R_3/\mathbf{m}_3$.

Example 6.14. Let (R, \mathbf{m}, k) be a 3-dimensional regular local ring with maximal ideal $\mathbf{m} = (x, y, z)R$. Consider the following sequence of local quadratic transforms

$$R := R_0 \subset R_1 := {}^x R_1 \subset R_2 := {}^{yx} R_2 \subset R_3 := {}^{zyx} R_3$$

defined by

$$\begin{aligned} S_1 &:= R\left[\frac{\mathbf{m}}{x}\right], & N_1 &:= (x, \frac{y}{x}, \frac{z}{x})S_1, & R_1 &:= (S_1)_{N_1}, & \mathbf{m}_1 &:= N_1 R_1 \\ S_2 &:= R_1\left[\frac{\mathbf{m}_1}{y/x}\right], & N_2 &:= (\frac{x^2}{y}, \frac{y}{x}, \frac{z}{y})S_2, & R_2 &:= (S_2)_{N_2}, & \mathbf{m}_2 &:= N_2 R_2 \\ S_3 &:= R_2\left[\frac{\mathbf{m}_2}{z/y}\right], & N_3 &:= (\frac{x^2}{z}, \frac{y^2}{xz}, \frac{z}{y})S_3, & R_3 &:= (S_3)_{N_3}, & \mathbf{m}_3 &:= N_3 R_3. \end{aligned}$$

Then:

- (1) Let $w_3 := \text{ord}_{R_3}$. Then $w_3(x) = 4$, $w_3(y) = 6$, $w_3(z) = 7$, and the images of $\frac{x^2y}{z^2}, \frac{y^3}{xz^2}$ in the residue field k_{w_3} of W_3 are algebraically independent over $R_3/\mathbf{m}_3 = k$.
- (2) The special $*$ -simple complete \mathbf{m} -primary ideal $P_{R_0 R_3}$ is a w_3 -ideal. We have
$$\begin{aligned} P_{R_0 R_3} &= \{\alpha \in \mathbf{m} \mid w_3(\alpha) \geq 18\} \\ &= (x^3y, y^3, xz^2, y^2z, x^3z, x^5, yz^2, x^2y^2, z^3, x^2yz)R. \end{aligned}$$
- (3) $\mathcal{BP}(P_{R_0 R_3}) = \{R_0, R_1, R_2, R_3\}$.
- (4) $\mathcal{B}(P_{R_0 R_3}) = \{3, 2, 1, 1\}$.
- (5) The set of Rees valuations of $P_{R_0 R_3}$ is $\{\text{ord}_{R_0}, \text{ord}_{R_1}, \text{ord}_{R_3}\}$.
- (6) Let $I := P_{R_0 R_3}$. Then :

- (a) $\text{Min}(\mathbf{m} R[It]) = \{Q_0, Q_1, Q_3\}$, where

$$Q_3 = (\mathbf{m}, y^2zt, x^3zt, x^5t, yz^2t, x^2y^2t, z^3t, x^2yzt)R[It]$$

$$Q_1 = (\mathbf{m}, y^3t, y^2zt, yz^2t, x^2y^2t, z^3t, x^2yzt)R[It]$$

$$Q_0 = (\mathbf{m}, x^3yt, x^3zt, x^5t, x^2y^2t, x^2yzt)R[It].$$

- (b) Let W_i denote the valuation rings corresponding to $w_i := \text{ord}_{R_i}$, where $W_i = R[It]_{Q_i} \cap \mathcal{Q}(R)$ for $i = 0, 1, 3$. Then $\frac{R[It]}{Q_3}$ is a polynomial ring in 3-variables over R/\mathbf{m} , and $\frac{R[It]}{Q_1} \cong \frac{(R/\mathbf{m})[T_1, T_2, T_3, T_4]}{(T_2 T_4 - T_3^2)}$ is a 3-dimensional normal Cohen-Macaulay domain with minimal multiplicity at its maximal homogeneous ideal with this multiplicity being 2. $\frac{R[It]}{Q_0} \cong \frac{(R/\mathbf{m})[T_1, T_2, T_3, T_4, T_5]}{\mathcal{J}}$ is a 3-dimensional normal Cohen-Macaulay domain with minimal multiplicity at its maximal homogeneous ideal with this multiplicity being 3, where the ideal \mathcal{J} is generated by the 2×2 minors of the matrix $\begin{bmatrix} T_1 & T_3 & T_4 \\ T_3 & T_4 & T_5 \end{bmatrix}$.

Proof. (1) : Since $\mathbf{m}_3 = (\frac{x^2}{z}, \frac{y^2}{xz}, \frac{z}{y})R_3$, we have $w_3(\frac{x^2}{z}) = w_3(\frac{y^2}{xz}) = w_3(\frac{z}{y}) = 1$, and hence $w_3(x) = 4, w_3(y) = 6, w_3(z) = 7$, and the images of $\frac{x^2 y}{z^2}, \frac{y^3}{xz^2}$ in the residue field k_{w_3} of W_3 are algebraically independent over $R_3/\mathbf{m}_3 = k$.

(2), (3), and (4) : The transform in R_3 of the ideal $K := (x^3 y, y^3, xz^2)R$ is the maximal ideal $\mathbf{m}_3 = (\frac{x^2}{z}, \frac{y^2}{xz}, \frac{z}{y})R_3$ and $w_3(K) = 18$. Let

$$I := \{\alpha \in \mathbf{m} \mid w_3(\alpha) \geq 18\} = (x^3 y, y^3, xz^2, y^2 z, x^3 z, x^5, yz^2, x^2 y^2, z^3, x^2 yz)R.$$

The ideal I is a w_3 -ideal. We see by direct computation that:

- (a) $I^{R_1} = ((\frac{y}{x})^3, y, (\frac{z}{x})^2, (\frac{y}{x})^2(\frac{z}{x}), z, x^2)R_1 = P_{R_1 R_3}$.
- (b) $I^{R_2} = K^{R_2} = (\frac{y}{x}, \frac{x^2}{y}, (\frac{z}{y})^2)R_2 = P_{R_2 R_3}$
- (c) $I^{R_3} = \mathbf{m}_3$.
- (d) $\mathcal{BP}(I) = \{R_0, R_1, R_2, R_3\}$.

Thus by [L, Proposition 2.1], we have $I = P_{R_0 R_3}$. It is clear that $\text{ord}_{R_0}(I) = 3, \text{ord}_{R_1}(I^{R_1}) = 2, \text{ord}_{R_2}(I^{R_2}) = 1, \text{ord}_{R_3}(I^{R_3}) = 1$. Hence $\mathcal{B}(I) = \{3, 2, 1, 1\}$.

(5) : By item (1), $w_3 := \text{ord}_{R_3}$ is a Rees valuation of $P_{R_0 R_3}$. We have the following table :

$P_{R_0 R_3}$	$x^3 y$	y^3	xz^2	$y^2 z$	$x^3 z$	x^5	yz^2	$x^2 y^2$	z^3	$x^2 yz$
$w_3 := \text{ord}_{R_3}$	18	18	18	19	19	20	20	20	21	21
$w_2 := \text{ord}_{R_2}$	9	9	10	10	10	10	11	10	12	11
$w_1 := \text{ord}_{R_1}$	5	6	5	6	5	5	6	6	6	6
$w_0 := \text{ord}_{R_0}$	4	3	3	3	4	5	3	4	3	4

Since the images of $\frac{x^3 z}{x^3 y}, \frac{x^3 z}{x^5}$ in the residue field k_{w_1} of W_1 are algebraically independent over k , $w_1 := \text{ord}_{R_1}$ is a Rees valuation of $P_{R_0 R_3}$, and also since the images of $\frac{xz^2}{z^3}, \frac{y^2 z}{y^3}$ in the residue field k_{w_0} of W_0 are algebraically independent over k , $w_0 := \text{ord}_{R_0}$ is a Rees valuation of $P_{R_0 R_3}$. Using Proposition 6.4, we conclude that $P_{R_0 R_3}$ has the three Rees valuations, $\text{ord}_{R_0}, \text{ord}_{R_1}$, and ord_{R_3} .

(6) : The statements of item (6) follow from the previous items and the connection between the Rees valuations of I and minimal primes of $\mathfrak{m} R[It]$ in the Rees algebra $R[It]$, cf. [HK]. \square

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907 U.S.A.

E-mail address: heinzer@math.purdue.edu

DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, JANGANGU SUWON 440-746, KOREA

E-mail address: mkkim@skku.edu